Math 20550 - Calculus III - Summer 2014
July 3, 2014
Exam 2

Name: ____

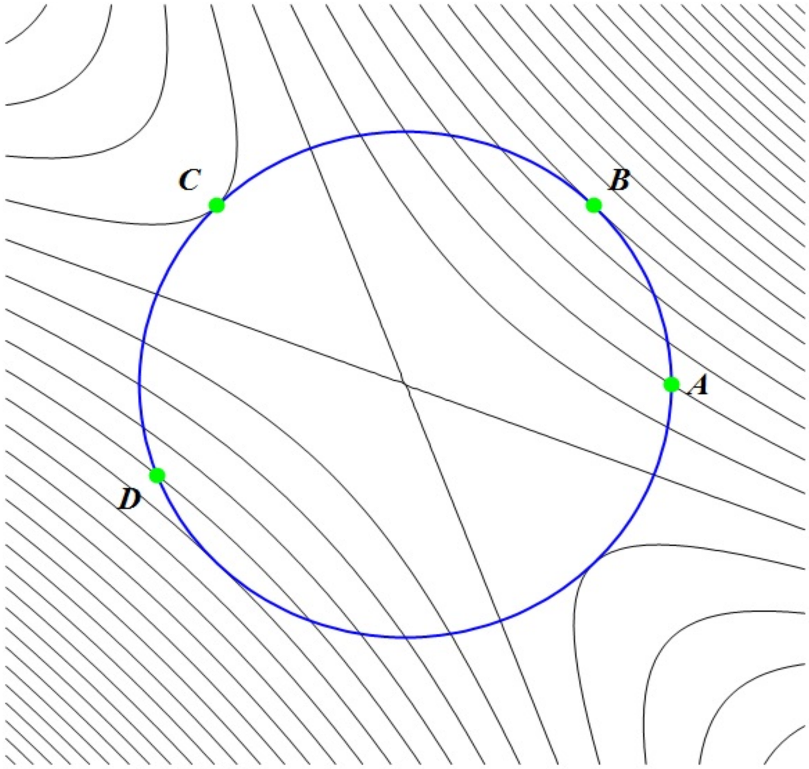
There is no need to use calculators on this exam. This exam consists of 12 problems on 12 pages. You have 75 minutes to work on the exam. There are a total of 105 available points and a perfect score on the exam is 100 points. All electronic devices should be turned off and put away. The only things you are allowed to have are: a writing utensil(s) (pencil preferred), an eraser, your 3×5 notecard, and an exam. No notes (other than the aforementioned notecard), books, or any other kind of aid are allowed. All answers should be given as exact, closed form numbers as opposed to decimal approximations (i.e., π as opposed to 3.14159265358979...). **You must show all of your work to receive credit. Please box your final answers.** Cheating is strictly forbidden. Good luck!

Honor Pledge: As a member of the Notre Dame community, I will not participate in, nor tolorate academic dishonesty. My signature here binds me to the Notre Dame Honor Code:

Signature:____

Problem	Score
1	/5
2	/15
3	/10
4	/5
5	/10
6	/5
7	/10
8	/10
9	/10
10	/10
11	/5
12	/10
Score	/100

Problem 1 (5 points)**.** *Consider the following contour plot for a function $f(x, y)$:*



The circle is a level curve $g(x, y) = k$. Which of the following must ALWAYS be true?

I. Subject to $g(x, y) = k$, $f(x, y)$ has a possible extremum at C.

II. Subject to $g(x, y) = k$, $f(x, y)$ has a possible maximum at A.

III. Subject to $g(x, y) = k$, $f(x, y)$ has a possible minimum at D.

IV. Subject to $g(x, y) = k$, $f(x, y)$ has an absolute maximum at B.

V. $f(x, y)$ has an absolute maximum or absolute minimum at C.

Recall that the Lagrange Multiplier theorem says that the gradients of f and g must be parallel at extrema of f subject to $g = k$. Observing the picture, the only marked point where this is true is C (recall that gradients are perpendicular to level curves).

Answer: I

Problem 2 (15 points - 5 points each)**.** *Consider the curve parametrized by*

$$\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle, \quad 0 \le t \le 2.$$

(a) *Find the arc length of the curve.*
(b) *Find the curvature at the point* $\left(2, 1, \frac{1}{3}\right)$.
(c) *Find an equation for the osculating plane at the point* $\left(2, 1, \frac{1}{3}\right)$.

Solution.

(a) $\mathbf{r}'(t) = \langle 2, 2t, t^2 \rangle$, so

$$\|\mathbf{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2 + t^2)^2} = 2 + t^2$$

Thus the arc length is

$$L = \int_0^2 (2 + t^2) dt = \left(2t + \frac{1}{3}t^3\right)\Big|_0^2 = 4 + \frac{8}{3} = \frac{20}{3}.$$

(b) First, we find the t-value at which $\mathbf{r}(t)$ passes though $\left(2, 1, \frac{1}{3}\right)$. Setting $\mathbf{r}(t) = \left\langle 2, 1, \frac{1}{3}\right\rangle$, the $\mathbf{i}$-component gives us that $t = 1$ and this checks with the other two components. Now $\mathbf{r}''(t) = \langle 0, 2, 2t \rangle$ and so

$$\mathbf{r}'(1) = \langle 2, 2, 1 \rangle \quad \text{and} \quad \mathbf{r}''(1) = \langle 0, 2, 2 \rangle$$

giving

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \langle 2, -4, 4 \rangle.$$

Thus, we have that the curvature at $\left(2, 1, \frac{1}{3}\right)$ (when $t = 1$) is

$$\kappa(1) = \frac{\|\mathbf{r}'(1) \times \mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|^3} = \frac{\sqrt{4 + 16 + 16}}{3^3} = \frac{6}{27} = \frac{2}{9}.$$

(c) The osculating plane at $\left(2, 1, \frac{1}{3}\right)$ uses $\mathbf{B}(1)$ as a normal vector, so we can use any parallel vector to $\mathbf{B}(1)$, namely, $\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 2, -4, 4 \rangle$. Thus, an equation for the osculating plane is

$$\langle 2, -4, 4 \rangle \cdot \left\langle x - 2, y - 1, z - \frac{1}{3} \right\rangle = 0$$

or

$$2x - 4y + 4z = \frac{4}{3}.$$

□

Problem 3 (10 points). *Let $z = f(x, y)$, where f is a differentiable function and suppose that $x = x(s, t)$ and $y = y(s, t)$. Given that*

$$x(1,0) = 2, \quad x_s(1,0) = -2, \quad x_t(1,0) = 6, \quad f_x(2,3) = -1$$

$$y(1,0) = 3, \quad y_s(1,0) = 5, \quad y_t(1,0) = 4, \quad f_y(2,3) = 10$$

Find $\dfrac{\partial z}{\partial t}(1,0)$.

Solution.

$$\begin{aligned} \frac{\partial z}{\partial t}(1,0) &= \frac{\partial z}{\partial x}(x(1,0), y(1,0))\frac{\partial x}{\partial t}(1,0) + \frac{\partial z}{\partial y}(x(1,0), y(1,0))\frac{\partial y}{\partial t}(1,0) \\ &= \frac{\partial z}{\partial x}(2,3)\frac{\partial x}{\partial t}(1,0) + \frac{\partial z}{\partial y}(2,3)\frac{\partial y}{\partial t}(1,0) \\ &= (-1)(6) + (10)(4) = -6 + 40 = 34 \end{aligned}$$

□

Problem 4 (5 points). *Compute* $\int \mathbf{r}(t)dt$ *where*

$$\mathbf{r}(t) = t\mathbf{j} + e^t\mathbf{k}.$$

Solution.

$$\begin{aligned} \int \mathbf{r}(t)dt &= \left(\int 0\,dt\right)\mathbf{i} + \left(\int t\,dt\right)\mathbf{j} + \left(\int e^t dt\right)\mathbf{k} \\ &= C_1\mathbf{i} + \left(\frac{1}{2}t^2 + C_2\right)\mathbf{j} + \left(e^t + C_3\right)\mathbf{k} \end{aligned}$$

□

Problem 5 (10 points)**.** *Suppose z is defined implicitly by*

$$z = e^x \sin(y+z).$$

Find all first partials of z.

Solution. This is *IMPLICIT* differentiation! Take $\dfrac{\partial}{\partial x}$ of both sides:

$$\frac{\partial z}{\partial x} = e^x \sin(y+z) + e^x \cos(y+z)\frac{\partial z}{\partial x} \qquad \text{product rule on the right side}$$

Solving for $\dfrac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x} - e^x \cos(y+z)\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}\left(1 - e^x \cos(y+z)\right) = e^x \sin(y+z)$$

So

$$\frac{\partial z}{\partial x} = \frac{e^x \sin(y+z)}{1 - e^x \cos(y+z)}.$$

Now, we do the like for finding $\dfrac{\partial z}{\partial y}$. Take $\dfrac{\partial}{\partial y}$ of both sides:

$$\frac{\partial z}{\partial y} = e^x \cos(y+z)\left(1 + \frac{\partial z}{\partial y}\right)$$

Solving for $\dfrac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial y} - e^x \cos(y+z)\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}\left(1 - e^x \cos(y+z)\right) = e^x \cos(y+z)$$

So

$$\frac{\partial z}{\partial x} = \frac{e^x \cos(y+z)}{1 - e^x \cos(y+z)}.$$

We can use the method from section 14.5 for doing implicit differentiation by first finding a function $F(x,y,z) = 0$ defining z implicitly as a function of x and y. In this case, such a function is

$$F(x,y,z) = e^x \sin(y+z) - z$$

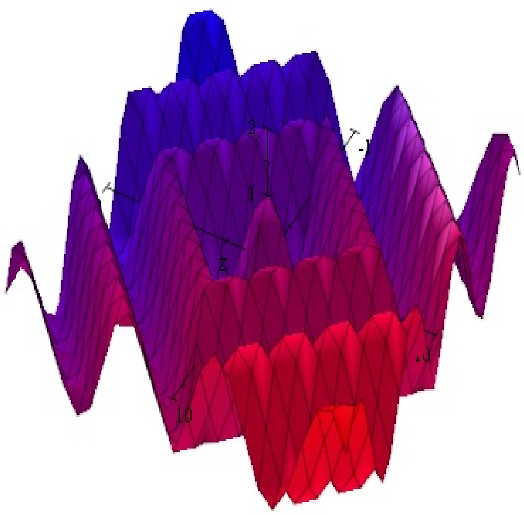
since $F(x,y,z) = 0$ gives us back the equation above. Then, we have that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

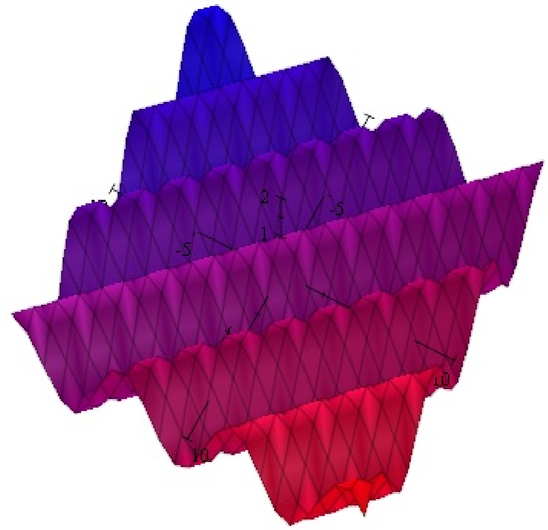
□

Problem 6 (5 points). *Which of the following is the graph of*

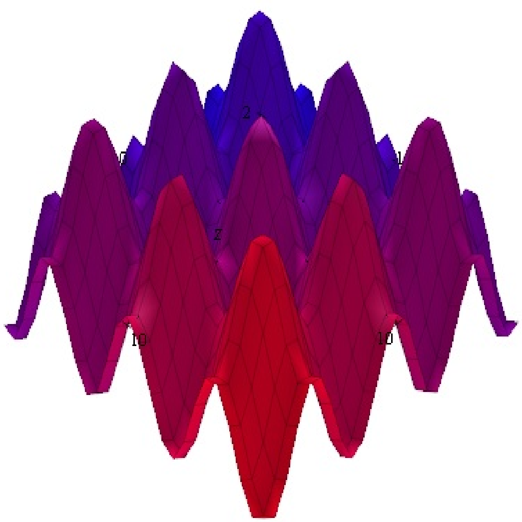
$$f(x, y) = \cos(|x| + |y|)?$$



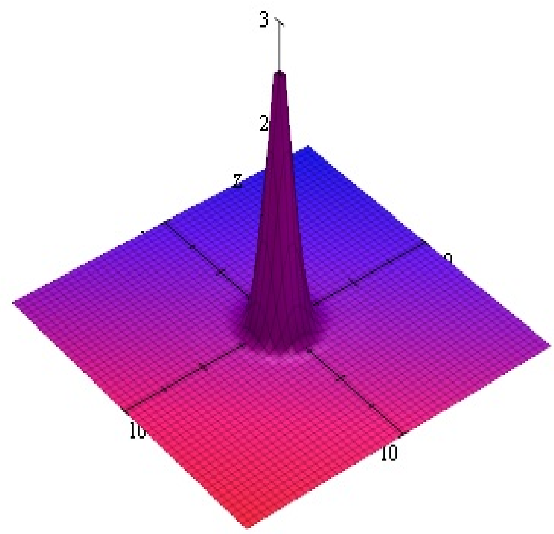
I.



II.



III.



IV.

The easiest way to see what this looks like is to see how the graph of $g(x, y) = |x| + |y|$ looks. The graph of $g(x, y)$ basically looks like an inverted pyramid with its point touching the origin. So, level curves of it are squares. This means $f(x, y)$ will looks like square waves expanding from the origin. (Like a ripple from dropping a stone into a pond, but instead of circular waves, square shaped waves.)

Ans:______I______

Problem 7 (10 points - 5 points each)**.** *Consider the function* $f(x,y) = x\tan y$.

(a) *Compute the derivative of* f *at* $\left(2, \frac{\pi}{4}\right)$ *in the direction of* $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

(b) *In which direction from* $\left(2, \frac{\pi}{4}\right)$ *is* f *decreasing the fastest?*

Solution.

(a) First, we check if $\mathbf{v}$ is a unit vector:

$$\|\mathbf{v}\| = \sqrt{1+1} = \sqrt{2}$$

Thus we need the unit vector

$$\widehat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

Now $\nabla f = \langle \tan y, x\sec^2 y\rangle$, so

$$\nabla f\left(2, \frac{\pi}{4}\right) = \left\langle \tan\frac{\pi}{4}, 2\sec^2\frac{\pi}{4}\right\rangle = \left\langle 1, 2(\sqrt{2})^2\right\rangle = \langle 1, 4\rangle\,.$$

Thus

$$D_{\widehat{\mathbf{v}}} f\left(2, \frac{\pi}{4}\right) = \nabla f\left(2, \frac{\pi}{4}\right)\cdot\widehat{\mathbf{v}} = \langle 1, 4\rangle\cdot\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = \frac{5}{\sqrt{2}}$$

(b) The direction of fastest decrease is in the direction of $-\nabla f = \langle -1, -4\rangle$.

□

Problem 8 (10 points)**.** *Find three positive numbers whose sum is* 15 *and the sum of whose squares is as small as possible.*

Solution. Let's call our numbers p,q, and r. Then, this problem asks to minimize the function

$$f(p, q, r) = p^2 + q^2 + r^2$$

subject to the constraint

$$p + q + r = 15.$$

Let's call the constraint function $g(p, q, r) = p + q + r$, then

$$\nabla f = \langle 2p, 2q, 2r \rangle \quad \text{and} \quad \nabla g = \langle 1, 1, 1 \rangle .$$

Then, the Lagrange multipliers method gives us the system of equations

$$\begin{cases} 2p = \lambda & \textcircled{1} \\ 2q = \lambda & \textcircled{2} \\ 2r = \lambda & \textcircled{3} \\ p + q + r = 15 & \textcircled{4} \end{cases}$$

Using equations ①, ②, and ③, we can get

$$\frac{\lambda}{2} = p = q = r$$

and plugging this information into ④ gives

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} = \frac{3\lambda}{2} = 15$$

meaning $\lambda = 10$. This gives that $p = q = r = 5$. Now, we need only check if $(5, 5, 5)$ is indeed a minimum of f. First, $f(5, 5, 5) = 25 + 25 + 25 = 75$. Now, another point on the constraint is $(6, 5, 4)$ and $f(6, 5, 4) = 36 + 25 + 16 = 77 > 75$, so $(5, 5, 5)$ is indeed a minimum. So, the three numbers are all 5.

One could also solve this problem by using the constraint to eliminate a variable from f, e.g., the constraint gives $p = 15 - q - r$, so we could rewrite $f(q, r) = (15 - q - r)^2 + q^2 + r^2$, then do the second derivatives test on this. This takes more effort, however.

□

Problem 9 (10 points)**.** *Find the tangential and normal components of acceleration for a particle traveling along the trajectory*

$$\mathbf{r}(t) = (3t - t^3)\mathbf{i} + 3t^2\mathbf{j}.$$

Solution. We start by computing $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$:

$$\begin{aligned} \mathbf{v}(t) = \mathbf{r}'(t) &= (3 - 3t^2)\mathbf{i} + 6t\mathbf{j} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= (-6t)\mathbf{i} + 6\mathbf{j} \end{aligned}$$

First, $a_T = \frac{d}{dt}\|\mathbf{v}(t)\| = \frac{d}{dt}\nu(t)$ (where $\nu(t) = \|\mathbf{v}(t)\|$ was my notation for *speed* in class). So, since $\|\mathbf{v}(t)\| = \sqrt{(3-3t^2)^2 + (6t)^2} = \sqrt{9 - 18t^2 + 9t^4 + 36t^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{(3+3t^2)^2} = 3 + 3t^2$ we have that

$$a_T = \frac{d}{dt}\|\mathbf{v}(t)\| = 6t.$$

Now, we know $\|\mathbf{a}(t)\|^2 = a_T^2 + a_N^2$, and

$$\|\mathbf{a}(t)\| = \sqrt{(-6t)^2 + 6^2} = \sqrt{36t^2 + 36}$$

so

$$a_N^2 = \|\mathbf{a}(t)\|^2 - a_T^2 = 36t^2 + 36 - 36t^2 = 36.$$

This gives $a_N = 6$ (a_N is always positive). □

Problem 10 (10 points)**.** *Find an equation for the tangent plane to* $x^4 + y^4 + z^4 = 3x^2y^2z^2$ *at the point* $(1, 1, 1)$.

Solution. First we realize this as a level surface of some function: $f(x, y, z) = 0$. We can take

$$f(x, y, z) = x^4 + y^4 + z^4 - 3x^2y^2z^2.$$

Then, the normal vector to this plane is $\nabla f(1, 1, 1)$:

$$\nabla f = \left\langle 4x^3 - 6xy^2z^2, 4y^3 - 6x^2yz^2, 4z^3 - 6x^2y^2z \right\rangle$$

so

$$\nabla f(1, 1, 1) = \langle -2, -2, -2 \rangle .$$

Thus, the equation for the tangent plane is

$$\nabla f(1, 1, 1) \cdot \langle x - 1, y - 1, z - 1 \rangle = \langle -2, -2, -2 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 0$$

or

$$-2x - 2y - 2z = -6$$

or

$$x + y + z = 3.$$

□

Problem 11 (5 points)**.** *Sketch the curve represented by*

$$\mathbf{r}(\theta) = \langle 3\sec\theta, 2\tan\theta \rangle .$$

Make sure to indicate the orientation of the curve (direction of increasing θ).

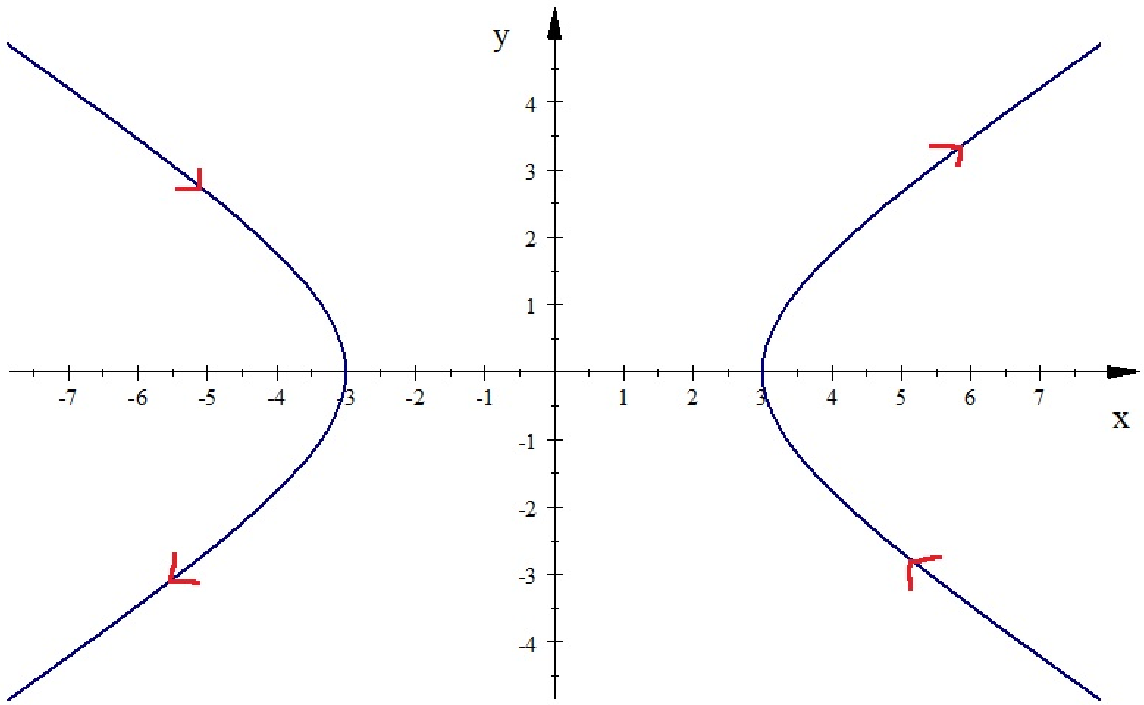
Solution. Recall that $\tan^2\theta + 1 = \sec^2\theta$ (which you can get by dividing the equation $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$), so that the curve $\mathbf{r}(\theta)$ satisfies the equation

$$\frac{y^2}{4} + 1 = \frac{x^2}{9}$$

or

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

which is recognizable as the equation of a hyperbola. Below is the plot with the orientation indicated:



□

Problem 12 (10 points)**.** *A particle with mass 2 starts at the origin with an initial velocity* $\mathbf{v}_0 = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$*. Its acceleration is* $\mathbf{a}(t) = 6t\mathbf{i} + 12t^2\mathbf{j} - 6t\mathbf{k}$*. Find its position and momentum functions. (Recall that its momentum,* $\mathbf{p}(t)$*, is mass times velocity.)*

Solution.

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt = (3t^2 + C_1)\mathbf{i} + (4t^3 + C_2)\mathbf{j} + (-3t^2 + C_3)\mathbf{k}$$

$$\mathbf{v}(0) = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

so

$$C_1 = 1, C_2 = -1, C_3 = 3$$

giving

$$\mathbf{v}(t) = (3t^2 + 1)\mathbf{i} + (4t^3 - 1)\mathbf{j} + (-3t^2 + 3)\mathbf{k}$$

Then

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = (t^3 + t + D_1)\mathbf{i} + (t^4 - t + D_2)\mathbf{j} + (-t^3 + 3t + D_3)\mathbf{k}$$

$$\mathbf{r}(0) = D_1\mathbf{i} + D_2\mathbf{j} + D_3\mathbf{k} = \mathbf{0}$$

so

$$D_1 = D_2 = D_3 = 0$$

giving

$$\mathbf{r}(t) = (t^3 + t)\mathbf{i} + (t^4 - t)\mathbf{j} + (-t^3 + 3t)\mathbf{k}.$$

So, the position and momentum functions are:

$$\mathbf{r}(t) = (t^3 + t)\mathbf{i} + (t^4 - t)\mathbf{j} + (-t^3 + 3t)\mathbf{k}$$

and

$$\mathbf{p}(t) = m\mathbf{v}(t) = 2\mathbf{v}(t) = (6t^2 + 2)\mathbf{i} + (8t^3 - 2)\mathbf{j} + (-6t^2 + 6)\mathbf{k}$$

□